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Non-thesis Master´s* Level Pre-Service Mathematics Teachers' Conceptions of Proof

As Concepções de Professores Pós-graduandos sobre Provas Formais

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Abstract

This case study research was carried out with eight pre-service teachers enrolled in a non-thesis Masters degree program at the university where the author works after having earned undergraduate degrees in mathematics from different universities in Turkey. The study is part of a large-scale study. The main part of the study aimed to identify the conceptions of participants about proof and proving in a period of ten weeks. The present study contains the preliminary findings regarding the participants' opinions about the meaning of proof and proving and the purposes of proof. Three groups of data were used in this article. The first group involves the essay writing of pre-service

* *Non-Thesis Master Program*: In Turkey there used to be (1997-2009) two alternative ways to become a mathematics teacher in high school level (grades 9-12). One of them was to graduate from the secondary level mathematics education departments (a five-year education) in the education faculties, and the other one was to graduate from the mathematics departments (a four-year education) in the faculties of science or faculties of science and letter. In order to attend these two departments (mathematics education and mahematics) mentioned, one had to pass a national exam called university entrance exam. The education faculty graduates took another national exam (KPSS) and they were appointed by Ministery of Education according to the points they took and their preferences. However, the gradutes of the faculties of science or faculties of science and letter had to complete a three-semester 'non-thesis master programe' (NtMP) to deserve the right to be appointed in addition to taking KPSS. NtMP which was available at some universities had limited number of vacancies. There were two important criteria to be accepted to this program, the grade point average and ALES (The Academic Stuff and Postgraduate Education Enterance Exam) point. NtMP was put into practice in 1997 by the Council of Higher Education (YÖK). The pre-service teachers were mainly presented to pedagogical knowledge issues and had the opportunity of practicing in high schools. The program was applied for the last time in 2009 in this form.

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teachers; the second group involves whole group discussions; and the last group contains individual semi-structured interviews. The results demonstrate that the pre-service teachers often prefer using formal discourse to define proof but have certain difficulties in making sense of these definitions. The general opinion of the participants about the purposes of proof concentrates on verification and explanation. Another problem examined in the study was concerned with whether the pre-service teachers' opinions about the meaning and purposes of proof can change. The results showed that their opinions may change regarding both. However, the changes involved expansion of their previous opinions by adding new dimensions, without moving in another direction.

Keywords: Proof. Proving. Conceptions of Proof. Pre-Service Mathematics Teacher.

Resumo

Este estudo de caso, realizado com professores pós-graduandos em cursos universitários nos quais atua o autor, na Turquia, é parte de uma pesquisa de grande escala. O objetivo principal do estudo é identificar as concepções desses pós-graduandos sobre a prova formal e a atividade de implementá-la. Este artigo explicita os resultados preliminares da pesquisa e centra-se na opinião dos participantes sobre o significado de prova, da ação de provar e das intenções dessa estratégia formal e de seu uso. Três grupos de dados foram utilizados neste artigo: o primeiro envolve ensaios escritos produzidos pelos pós-graduandos; o segundo envolve as discussões em grupo e o último grupo refere-se a entrevistas individuais semi-estruturadas. Os resultados demonstram que os professores muitas vezes preferem o discurso formal para definir a prova, mas têm dificuldades quando solicitados a atribuir significado a essas definições. A opinião geral dos participantes sobre os objetivos de uma prova formal foca-se na verificação e na explicação. Outro objetivo do estudo foi atentar para a possibilidade de alteração das concepções desses professores pós-graduandos sobre os temas em foco. Nesse sentido, os resultados mostram que as suas opiniões, tanto em relação ao significado quanto aos objetivos de uma prova, se alteram. Tais mudanças envolvem novas ideias acerca das concepções anteriormente defendidas e incorporam novas dimensões, mas não são suficientes para alterar significativamente as crenças prévias.

Palavras-chave: Provas Formais. Concepções sobre Provas. Formação Continuada.

1 Introduction

Proofs are among the key instruments in the development of mathematics even to the present day. Proofs are considered as the central element of mathematics and its teaching process (KO, 2010; HEINZE et al., 2008; HEINZE; REISS, 2003; BALL et al., 2002; KNUTH, 2002a; TALL, 1999). "Some argue that the game of mathematics is called proof; if there is no proof, then there is no mathematics" (DAVIS; HERSH, 2002: 174). From a broad perspective, proof is a basic activity in doing and understanding mathematics which warrants mathematical knowledge (ALMEIDA, 2000).

The characteristics attributed to proof until now, and its significance in a historical panorama, are closely related to the roles it plays in understanding, producing, sharing and transmitting mathematical knowledge, as a process as well as a product. One of the main functions of proof recognized in the field of mathematics teaching is to verify theorems, propositions or conjectures (SMITH, 2006; AVIGAD, 2005). However, Hanna (2000) argues that proofs not only serve to demonstrate whether mathematical statements are true or false, but also why they are true, a function which is more important for teaching proofs. Resnik (1992) notes that by using proof, mathematicians aim to provide alternative demonstrations of previous results (at times by a simpler or more economic demonstration than the previous one, and sometimes by using information obtained from a different area of mathematics), in addition to demonstrating new ones. Some studies have more systematically presented the functions of proofs. Drawing upon these studies, the functions of proof could be listed as verification, explanation, communication, discovery, systemization, intellectual challenge and construction of empirical theory (YACKEL; HANNA, 2003; KNUTH, 2002a; DE VILLIERS, 1999). It is hard to confine proof to a narrow definition since it has a broad functional scope and significance. In simplest terms, proof could be designated as an operation of deriving conclusions from conjectures. Despite this simple description, it is not at all easy to define proof at a conceptual level and to explain its nature (RAMAN, 2003). What proof is has long been a subject of debate among mathematicians, philosophers and educators (LEE, 2002; HEALY; HOYLES, 2000; HANNA, 2000). These debates originated both from the fact that proof is complex in itself - an activity that can be performed on the basis of a series of mental and logical processes in a variety of different ways - and from the rich diversity of its functions. Although rigorous proof in particular has a clearer conceptual structure upon which there is consensus in theoretical mathematics, there is no such consensus in the field of mathematics education (BALACHEFF, 2008), and this could be identified as one of the obstacles to the formation of a common notion of proof teaching.

Even though proofs occupy an important place in school and advanced mathematics, for various reasons, it is not an easy process to comprehend, perceive, and explore, and the importance of knowing how to carry out a proof in teaching is often overlooked. Despite such difficulty, proofs are indispensable for mathematics teaching at any level. In fact, the emphasis and interest related to proof in the field of mathematics education research area (HOFE et al., 2003) and movements to reform mathematics curricula have increased recently in many countries (e.g. USA (STYLIANIDES, 2009), Sweden, Italy and Estonia (REUTERSWARD; HEMMI, 2011)) .

For instance, the status of proof increased considerably compared to the previous standards of the National Council of Teachers of Mathematics [NCTM] for the year 2000 (KNUTH, 2002a), and this new approach particularly emphasizes the role and importance of proof as well as increasing its emphasis in mathematics teaching.

In the NCTM document for the year 2000, proof was designated as one of the four standards of basic processes; and teaching proof is deemed necessary for all levels from pre-kindergarten to the 12th grade. The document in question highlights the following points about reasoning and proof, which are summarized below:

Instructional programs from kindergarten through grade 12 should enable all students to

> -- Recognize reasoning and proof as fundamental aspects of mathematics
> -- Make and investigate mathematical conjectures
> -- Develop and evaluate mathematical arguments and proofs
> --Select and use various types or reasoning and methods of proof (NCTM, 2000, p. 56).

The NCTM principles and standards would naturally have certain impacts on teaching curricula. As a matter of fact, the general framework of the new mathematics curricula restructured in Turkey is consistent with the NCTM document.

As for the current state of the curricula with regard to the place and significance of proof, the following points can be made: When compared to the previous (1992) curriculum, the new secondary mathematics curriculum implemented today (the 2005 curriculum) involves some differences in the way proof and proving are treated. The introduction of the new mathematics curriculum deals with proof when presenting the fundamentals of mathematical study using the following statements:

> -- To find out and understand logical relationships,
> -- To categorize the relationships detected and to verify them,

-- To generalize and verify relationships and apply them to daily life (MNE, 2005, p. 4).

Furthermore, the curriculum designates proof as a sub-learning domain within the domain of logic learning included among the first subjects of grade 9, and uses a scheme with regard to the methods of proving. Following these observations about the mathematics curriculum, it could be concluded that the new curriculum attaches relatively greater importance to proof. A limited number of mathematics educators in Turkey studying proof and proving expect that the revised mathematics curricula will enhance the status of proof. The same expectation is likely to be found among other researchers as well. Knuth (2000, p.1) suggests the following: "Recent reform efforts in the United States are calling for substantial changes in both school mathematics curricula and teachers' instructional practices with respect to proof."

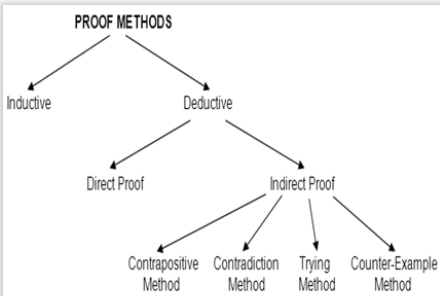


Figure 1 - Methods of proof (MNE, 2005, p. 25)

Today's teachers are mainly expected to gain insight into the notional changes about teaching proof within the framework of reform movements, and to educate themselves and later their students in line with this notion.

2 The role and importance of proof in mathematics teacher education

The abstractness of content is one of the basic points that distinguishes school mathematics (middle and secondary levels) and university-level mathematics. University mathematics is more abstract and theoretical than school mathematics, mainly because teaching focuses on theorems and proofs, a reason which also approximates university mathematics to academic mathematics. In essence, academic mathematics is the science of proving and a field of mathematics research. If we are to term university level and academic mathematics together as advanced mathematics, the following points could be made: 'Advanced mathematics is characterized by its emphasis on proof' (HOUSTON, 2010) and for every learner of mathematics in the field of advanced mathematics, proving is a highly important skill (WEBER, 2001). That is the reason why proving is one of basic skills required for mathematics teachers to acquire during their education. A group of main courses (Calculus, Algebra, Linear Algebra, Analytic Geometry, etc.) taught in the process of mathematics teacher training involves the structuring of subject matter knowledge and contains theoretical knowledge. Such courses often use verified theorems and proofs with different applications. The process of training mathematics teachers is carried out in many countries through undergraduate and even graduate education. For this reason, it is inevitable that pre-service teachers will come across proofs in courses in which they are expected to acquire content and pedagogical content knowledge. The content of their courses about subject matter serves to build and cognitively construct knowledge about advanced mathematics; in addition, courses related to pedagogical content knowledge are supposed to contribute to the process of teaching proofs to primary and secondary level students and help them acquire skills for proving. As a result, pre-service mathematics teachers are expected to have the necessary knowledge and skills regarding proofs, since they are mathematics students as well as future mathematics teachers. The relevant research has demonstrated that perceptions, attitudes, experience and skills of teachers about proofs influence the competence level of students in proving (GALBRAITH, 1995; KNUTH, 2002a); and thus, teachers play a critical role in the students' learning and understanding of proof. For this reason, there is a need to investigate the conceptions, knowledge and performance of both pre-service and in-service teachers about proof and proving. Knuth (2002a, p.63) says the following about this point: "research that examines teachers' conceptions of proof in the context of secondary school mathematics is greatly needed".

Based on this idea, the present study attempted to determine the conceptions of pre-service mathematics teachers enrolled in a non-thesis Masters degree program about proof and proving. The main research question was "*What are the conceptions about proof and proving of pre-service mathematics teachers in the non-thesis Master's degree program?*" Other sub-questions addressed on the basis of this main problematic are as follows:

- *How is proof defined by the participants?*
- *What are the opinions of the participants about the purposes/ functions of proof?*

- *Can these opinions change?*

3 Methodology

This is a qualitative case study in terms of its design, data collection, and data analysis processes. The study is a part of larger-scale study. In line with the qualitative research paradigm, multiple data collection was implemented in the study. The data were grouped in three, which are writing essays, group discussions, and semi-structured interviews.

3.1 Participants

Eight pre-service mathematics teachers participated in the study. They were all graduates of mathematics departments of five different public universities. Seven of the students studied at universities in western Turkey, while one studied at a university in the south. At the time when the study was conducted, all students were enrolled in the non-thesis Master's program in an education faculty (where the author also works) to be appointed as teachers as required by the current legislation in Turkey (this program was available for the last time in 2009 when the study was conducted, and was terminated the following year). Four of the participants are female and four are male. They had not taken any courses about proof and proving as part of the non-thesis Master's program, but voluntarily agreed to participate in the study upon the author's request. The pre-service teachers met the author after school hours at least once a week for 10 weeks, and responded to questions that aimed to identify their conceptions about proof and proving. All of the participants had graduated from mathematics departments with good grade point averages (GPA). The pre-service teacher graduating with the lowest GPA had 3.18 (out of 4.00), while the highest GPA was 3.65. Four of the participants had graduated from the same university, while each of the remaining four graduated from different universities. All participants had graduated within the last two years, and two of them had teaching experience after graduation, both of whom were female and worked as trainee mathematics teachers for one year at different private schools. The pre-service teachers were asked if they received any special training and read any books about proof and proving during or after their education. None of the pre-service teachers had received any special training, while only one of them stated that he had read a book about proving.

Names	Sex	University	GPA	Experience	Book
YELİZ	F	Ege University, Mathematics Depart.	3.30	--	--
ZEHRA	F	Balıkesir University, Mathematics Depart.	3.23	1 year	--
BANU	F	Ege University, Mathematics Depart.	3.65	1 year	--
SERAP	F	Anadolu University, Mathematics Depart.	3.25	--	--
ONUR	M	Afyon Kocatepe University, Mathematics Depart.	3.28	--	--
HAKAN	M	Ege University, Mathematics Depart.	3.18	--	--
ILKER	M	Çukurova University, Mathematics Depart.	3.53	--	X
ARDA	M	Ege University, Mathematics Depart.	3.33	--	--
Total 8	Rate %50	Total different 5 University, 4 of them from Ege Univ.	Mean 3.34	2	1

Table 1 - Some information about the pre-service teachers

Table 1 summarizes the information about the participants (The names that are used in the table are pseudonyms).

3.2 Data collection

-Writing Essays

The participants were presented a series of concepts about proof and proving (e.g. proof, theorem, verification, abstraction), and were asked to provide explanations about these. All the concepts were addressed in a single session, and the participants were given a total of two hours to explain them.

- Whole Group Discussions

Following written explanations, two group discussions were held with the pre-service teachers about the explanations they provided on paper. During the discussions, all participants were given the opportunity to share and discuss their thoughts in groups in an environment in which they all interacted and the researcher played the role of an unbiased motivator. Both discussion sessions were videotaped and transcribed.

- Semi-Structured Interviews

Following group discussions, semi-structured interviews were held in three individual sessions with the pre-service teachers. At each session, 6 or 7 questions were addressed to the participants. All interviews were audio-taped and all recordings were transcribed.

This study presents the preliminary results by using a part of all three data groups. It specifically focused on the opinions of participants about how proof and proving are defined, and what the main purposes of proof are.

3.3 Data analysis

During data analysis, the author first carried out multiple readings of all data groups. The main aim of this process was to identify how proof was perceived by the pre-service teachers as a product and a process, so the researcher focused on relevant sections in the explanations. The data were assessed as a whole, and attempts were made to describe the general perception of the participants with its components in different contexts. In the analysis, the results of the studies on learners' conceptions in the literature were examined, and discourse analysis was performed on the transcribed texts.

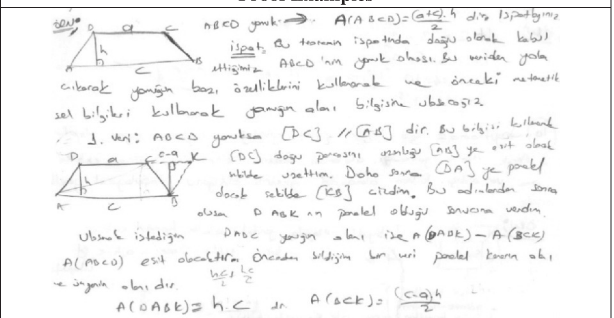
4 Results

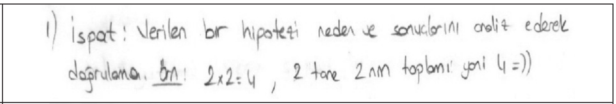
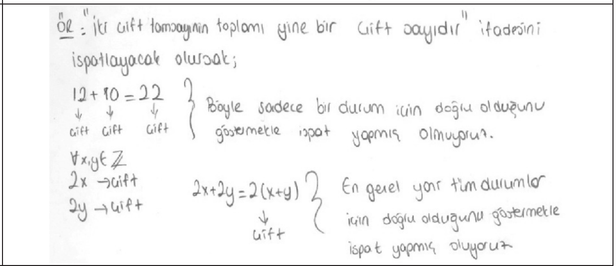
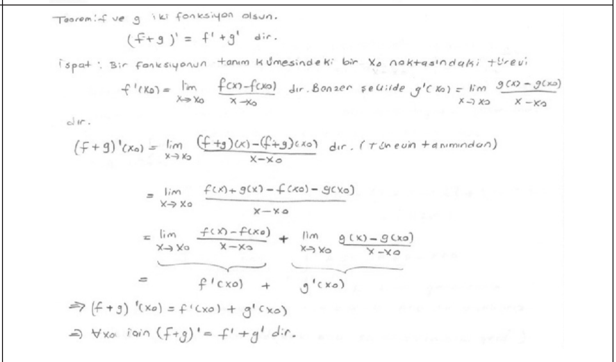
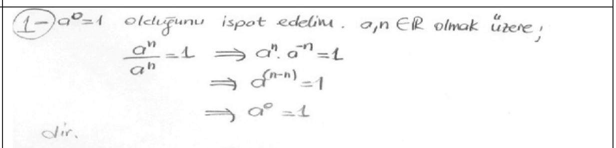
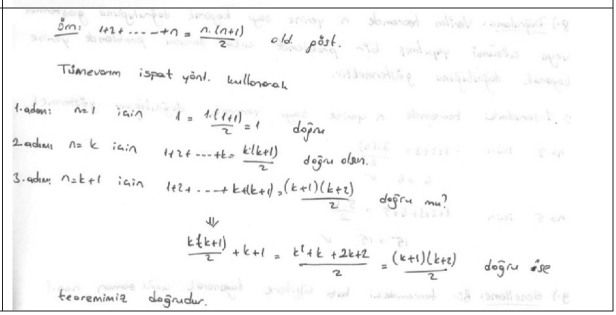
In the first data group, *proof* was among the chief concepts addressed to the pre-service teachers and that they were asked to write about. In the explanations they wrote as a response to the question "what do you think proof is?", nearly all of them tried to provide a formal definition, but not in a strict sense. What is meant by "formal" here can be considered to have a similar meaning to Knuth's definition of *less formal proof*[1]. Table 2 presents the written explanations and examples of proof by participants. The definitions of pre-service teachers are more or less similar, and from the common points in all of their responses, we could obtain the following primitive definition: "*Proof is the verification of a mathematical statement*".

An examination of the definitions shows that only Yeliz did not use the term 'verification', and defined proof as 'arriving at the desired solution by using the data in a theorem'. It is seen in the table that out of eight individuals, seven used the term 'verification' in their definitions, which gives a general idea about what the pre-service teachers think of the purposes of proof. The pre-service teachers defined proving as verifying a theorem, proposition or mathematical expression. In their definitions, while Yeliz did not directly use the term verification but implies it, Hakan and Arda stated that it was the 'theorem' that requires verification, while Zehra referred to 'hypothesis', Banu and Serap to 'statement', Onur to 'proposition', and Ilker to both proposition and theorem. Only two individuals partly deviated from the primitive definition. The first of these two was Yeliz, who was different from others in her description in that her approach

[1] Martin and Harel (1989) point out that *formal proofs* are very ritualistic in nature, tied heavily to prescribed formats and/or the use of particular language (cf. KNUTH, 2002a, p. 72). Less formal proofs do not necessarily have a rigidly defined structure or are not perceived as being "mathematically rigorous".

also referred to the mechanism of proof, and explained that the result is obtained step by step starting from the given data and by taking what is given in a theorem to be true. The other different definition was provided by Banu, who used in her explanation the statement that 'showing that a proposed statement is true for all existing cases on the basis of certain conjectures', by which she both mentioned conjectures and revealed what meaning she assigned to verification. There was no notable error or shortcoming in the proof definitions of (all) pre-service teachers. Certainly, this observation is true when we take proof in its simplest definition as "an operation of deriving conclusions from conjectures", as we noted in the introduction. In their written explanations, four pre-service teachers felt the need to provide additional information along with their definitions. Yeliz stated that proofs can be carried out not only to prove that something is true, but also to demonstrate what is false about something that is false; Banu wrote that while proving a statement, verification should be performed for all existing cases; and Serap and Onur noted the same point by arguing that there are certain methods of proof like detecting the contradictions. Among those who gave additional explanations, only Banu represented her explanations in her proof example. Banu noted that verification should be made for all cases and after a numerical example, she demonstrated that the sum of two even numbers is always even using the variables of x and y. The others did not have such a purpose and simply gave additional information that they found to be important. In the first data group, the aim was not to determine whether the pre-service teachers provided accurate and clear definitions or to select the best of all explanations. The aim was to identify the first (simple) idea that comes to their minds about proof and the other concepts. Subsequently, it was investigated using the other data groups to what extent these isolated-simple ideas made sense for them and whether the process varied. The pre-service teachers were asked to present an example on paper to embody their proof definitions.

	Expressions	**Proof Examples**
YELIZ	*Arriving at the solution required by the data in a theorem given in mathematics step by step starting from the given data and by taking what is given in the theorem to be true. Proving is not necessarily made for things that are true. One can prove what is false about [something] false.*	

ZEHRA	*To verify a given hypothesis by analyzing the reasons and results.*	 1) İspat: Verilen bir hipotezi neden ve sonuçlarını analiz ederek doğrulama. Ör: $2 \times 2 = 4$, 2 tane 2nin toplamı yani 4 =))
BANU	*Proof is to demonstrate that a proposed statement is true for all existing cases on the basis of certain conjectures. Verifying it for one or a few cases does not mean proving; we need to verify it for all existing cases.*	 Ör: "İki çift tamsayının toplamı yine bir çift sayıdır" ifadesini ispatlayacak olursak; $12 + 10 = 22$ (çift, çift, çift) } Böyle sadece bir durum için doğru olduğunu göstermekle ispat yapmış olmuyoruz. $\forall x, y \in \mathbb{Z}$ $2x \to$ çift $2y \to$ çift $2x + 2y = 2(x+y)$ → çift } En genel yani tüm durumlar için doğru olduğunu göstermekle ispat yapmış oluyoruz
SERAP	*Operations to verify a mathematical expression, proposition, or theorem. There are various proof techniques. For example, proof by induction and proof by contradiction, etc.*	 Teorem: f ve g iki fonksiyon olsun. $(f+g)' = f' + g'$ dir. İspat: Bir fonksiyonun tanım kümesindeki bir x_0 noktasındaki türevi $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ dir. Benzer şekilde $g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$ dir. $(f+g)'(x_0) = \lim_{x \to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0}$ dir. (Türevin tanımından) $= \lim_{x \to x_0} \frac{f(x) + g(x) - f(x_0) - g(x_0)}{x - x_0}$ $= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$ $= f'(x_0) + g'(x_0)$ $\Rightarrow (f+g)'(x_0) = f'(x_0) + g'(x_0)$ $\Rightarrow \forall x_0$ için $(f+g)' = f' + g'$ dir.
ONUR	*The method followed to verify any proposition. (Proving that the opposite of what is to be proven is false is also a proof method.)*	 1- $a^0 = 1$ olduğunu ispat edelim. $a, n \in \mathbb{R}$ olmak üzere; $\frac{a^n}{a^n} = 1 \Rightarrow a^n \cdot a^{-n} = 1$ $\Rightarrow a^{(n-n)} = 1$ $\Rightarrow a^0 = 1$ dir.
HAKAN	*To demonstrate that a given theorem is true by using mathematical methods.*	 Örn: $1 + 2 + \dots + n = \frac{n \cdot (n+1)}{2}$ old. göst. Tümevarım ispat yönt. kullanarak 1. adım: $n = 1$ için $1 = \frac{1 \cdot (1+1)}{2} = 1$ doğru 2. adım: $n = k$ için $1 + 2 + \dots + k = \frac{k(k+1)}{2}$ doğru olsun. 3. adım: $n = k+1$ için $1 + 2 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}$ doğru mu? ⇓ $\frac{k(k+1)}{2} + k + 1 = \frac{k^2 + k + 2k + 2}{2} = \frac{(k+1)(k+2)}{2}$ doğru ise teoremimiz doğrudur.

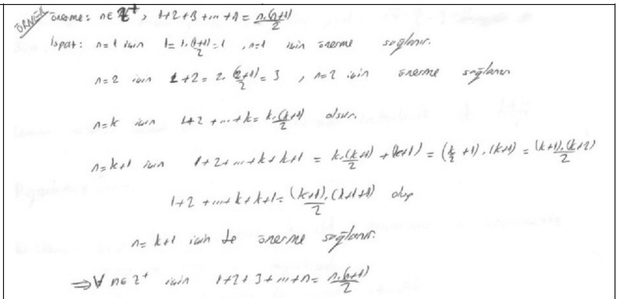
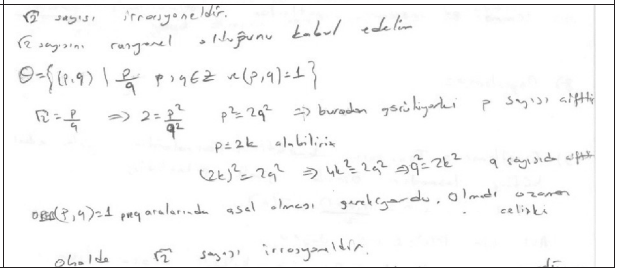
ILKER	*It is to prove that a given proposition or theorem is true by using known propositions or theorems.*	 ÖRNEK Önerme: $n \in Z^+$, $1+2+3+ ... +n = \frac{n.(n+1)}{2}$ İspat: n=1 için $1=\frac{1.(1+1)}{2}=1$, n=1 için önerme sağlanır. n=2 için $1+2=\frac{2.(2+1)}{2}=3$, n=2 için önerme sağlanır n=k için $1+2+...+k=\frac{k.(k+1)}{2}$ olsun. n=k+1 için $1+2+...+k+k+1 = \frac{k.(k+1)}{2}+(k+1) = (\frac{k}{2}+1).(k+1) = \frac{(k+1).(k+2)}{2}$ $1+2+...+k+k+1=\frac{(k+1).(k+1+1)}{2}$ olur n=k+1 için de önerme sağlanır. $\Rightarrow \forall n \in Z^+$ için $1+2+3+...+n=\frac{n.(n+1)}{2}$
ARDA	*It serves to demonstrate that a given theorem is true.*	 $\sqrt{2}$ sayısı irrasyoneldir. $\sqrt{2}$ sayısını rasyonel olduğunu kabul edelim $Q=\{(p,q) \mid \frac{p}{q}$ $p,q \in Z$ ve $(p,q)=1\}$ $\sqrt{2}=\frac{p}{q} \Rightarrow 2=\frac{p^2}{q^2}$ $p^2=2q^2$ ⇒ buradan görülüyor ki p sayısı çifttir p=2k alabiliriz $(2k)^2=2q^2 \Rightarrow 4k^2=2q^2 \Rightarrow q^2=2k^2$ q sayısı da çifttir OBEB(p,q)=1 p ve q aralarında asal olması gerekiyordu. Olmadı o zaman çelişki O halde $\sqrt{2}$ sayısı irrasyoneldir.

Table 2 - Responses and proof examples of pre-service teachers

Yeliz tried to demonstrate that the formula that gives the area of an ABCD trapezoid is $_s$(ABCD)=(a+c).h/2; Zehra used only a single numerical verification to show that 2x2 equals the sum of two twos; Banu demonstrated that the sum of two even integers is always even; Serap showed that the derivative of the sum of two functions is the sum of the derivatives of the two functions; Onur tried to demonstrate $a^0=1$; Hakan and Ilker showed that 1+2+3+ … +n=n(n+1)/2; and Arda demonstrated that $\sqrt{2}$ is irrational. Except for Serap, all pre-service teachers selected secondary-level examples.

Yeliz chose to demonstrate a geometrical formula and gave an acceptable proof by providing an additional drawing. Zehra's example is far from being a proof example and is simply a numerical example. Banu chose a theorem from the number theory and provided a proof that is correct in its reasoning, but weak in mathematical notation. Serap dealt with one of the main theorems in calculus and presented a common proof at undergraduate level.

Onur preferred a basic characteristic of exponential numbers, which he proved by a practical demonstration. In a different way, Hakan and Ilker preferred to use mathematical induction and their examples were identical. Hakan failed to show the last step of proof, while Ilker completed his proof (by using a more appropriate mathematical language than Hakan). Arda chose a widely known example that used deductive method and correctly demonstrated that $\sqrt{2}$ is irrational. All of the pre-service teachers used examples that were parallel to their proof definitions (and possibly those that first occurred to them) and except for Zehra, all of them followed more or less correct ways.

The second data group involves the whole group discussions held with the pre-service teachers. During the first discussion session, the essays were handed back to the pre-service teachers and the concepts they wrote about were discussed after reviewing their explanations. Table 3 presents an excerpt containing the discussions about what proof is.

1	**R**	How did you define proof? [**R**: Researcher]
2	**Zehra**	To verify a given hypothesis by analyzing the reasons and results.
3	**Ilker**	I don't think you can do a proof by analyzing the reasons and results.
4	**Zehra**	How do you do it?
5	**R**	Explain it a bit; tell us what you think louder.
6	**Ilker**	I mean if there are other verified propositions, you can use them. I mean by simply analyzing that particular example, I mean, that … I see what you mean but I think it is lacking. The definition is inadequate.
7	**Zehra**	So what would you like to add?
8	**Ilker**	I guess one should not deny the presence of previously known propositions.
9	**Zehra**	Ok, so can I not use previous propositions while investigating the reasons?
10	**Ilker**	That is what is lacking in the definition, it is not clear.
11	**R**	So he means implicit; or even if it exists, we can add it.
12	**Onur**	I wrote it is a method followed to verify.
13	**R**	To verify what?
14	**Onur**	Any claim.
15	**R**	Ok, your friend says that it is a rough definition and needs more details.
16	**Zehra**	Let us hear his definition. What did you say, Ilker?
17	**Onur**	Can we see it?
18	**Ilker**	I said it is to prove that a given proposition or theorem is true by using known propositions or theorems.
19	**Onur**	Do we have to do it by using what is known?
20	**Ilker**	How can we do it with anything unknown?
21	**Zehra**	Why? One cannot reason?
22	**Onur**	You mean a proposition, yes.
23	**Ilker**	So…
24	**Onur**	It helps all.
25	**Ilker**	Even if one reasons, he certainly has something he [previously] knows. He has to have a known thing. Otherwise, he cannot prove anything if he does not know anything.
26	**R**	What do we use to prove a theorem? What premises do we use?
27	**Yeliz**	The data.
28	**Banu**	We use conjectures.
29	**Ilker**	Other propositions
30	**Ilker**	We use verified theorems.
31	**Arda**	Lemmas.
32	**Zehra**	Induction.
33	**Ilker**	That is a method of proof.
34	**Zehra**	And you don't need anything for that.
35	**R**	Can you explain it?
36	**Zehra**	I mean induction does not require any hypothesis or previously verified proposition, theorem, or anything.
37	**Banu**	It was what I meant; we just use conjectures.
38	**Zehra**	Generally thinking, not every proof requires a proposition.
39	**Ilker**	But to use the proof, or to obtain that method of proof, I also used induction for proof. I only needed the method of proof in my proof. I did not need a known thing. But this only applies to the proof, so the model here was obtained by known facts?
40	**Onur**	And here you need to know something from the other group, so it does not come naturally.
41	**Zehra**	In fact, proof is not something easy. Proving requires a high-level skill, so one has to have some previous knowledge when we think about it.

Table 3 - Excerpt 1 (Whole group discussion)

At the beginning of the debate, a difference of opinion arose with the objection made against Zehra's definition (3). Ilker stated that the expression 'analyzing the reasons and results of a given hypothesis' given in the proof definition was inadequate and one should also add that this process involves the use of some previous information (8). Ilker presented his own definition (18) to make up for this shortcoming. At this point, Onur addressed an interesting question by asking whether proof should necessarily done using something known (19) and Zehra claimed that one can make proof simply by his/her own reasoning (21). In order to deepen the discussion, the researcher asked 'what do we use to prove a theorem?' (26).

To this question, the pre-service teachers responded with correct explanations. At this juncture, another interesting situation appeared and Zehra stated that one of the premises used in proving is "induction" (32). This misconception added another dimension to the discussion, resulting in the questioning of induction by the pre-service teachers. Zehra argued that induction does not require any proven proposition, theorem or any other thing (36), while Banu stated that only conjectures are used (37). Zehra expressed her opinion in the form of a judgment by saying "… not every proof requires a proposition …" (38). Ilker believed that what is required in induction is to know only the form of proof (the beginning and steps of induction) (39). Onur partially disagreed with this idea, (40) while Zehra stated that proving is a higher-order skill, so one needs to have some previous knowledge to carry out a proof (41). In the subsequent stages of the debate, the participants questioned whether induction is a method of proof (see Table 4) and some perceptions about proof were revealed at this stage.

The debate started when Arda asked "I still cannot figure out whether induction is a method of proof" (42). Ilker, Zehra, Serap and Banu agreed that induction is a method of proof. Ilker expressed his agreement about induction as a method of proof, but gave voice to a different interpretation by saying induction is the proof of proof (51). The researcher told the pre-service teachers that there is a direct relation between the meaning we assign to the definition of proof and deciding whether induction is a method of proof (54). Then Banu read her own definition of proof, noting that in proving, verification should be made for all existing cases (55). The following discussions were concerned with whether induction meets this condition.

Despite all discussions, the suspicions in Arda's mind did not fully disappear. At this point, although it is not a clear and correct explanation, Ilker emphasized that since the method used in induction has already been proven, what is done when using this method is the proof of proof (53, 61).

42	**Arda**	I still cannot figure out whether induction is a method of proof.
43	**Banu**	It is a method of proof.
44	**Ilker**	Yes.
45	**Arda**	For example, you say m=1 is true, then you take n equals as true. Then you show that it is also true for n=k+1. Is this a method of proof?
46	**Zehra**	Yes, it is a method of proof. We use it for many things.
47	**Serap**	No.
48	**R**	Who thinks that induction is not a method of proof?
49	**Arda**	We learned that it is a method of proof but I still cannot understand it.
50	**Yeliz**	There is a conjecture there as well. If we say conjecture, I mean, we first assume that it is equal to go on.
51	**Ilker**	I think, sir, the induction method; in fact, we prove a proof by using induction.
52	**R**	Prove a proof?
53	**Ilker**	Yes. Because induction was already verified. You use verified data to arrive at the same solution for different cases about a given proposition.
54	**R**	I think that depends on what we understand from the definition of proof. So take your definition of proof and include induction, then interpret.
55	**Banu**	I said the following for proof; to prove or demonstrate that a proposed statement or a proposition etc. is true for all existing cases on the basis of certain conjectures.
56	**R**	Then, according to this definition, is induction a method of proof?.
57	**Ilker**	Yes, after all, we also generalize it for all cases.
58	**Arda**	We do not demonstrate it for intermediate cases; for example, we do not show there intermediate cases.
59	**R**	Are you sure?
60	**Banu**	But you say k and verify it for k+1.
61	**Ilker**	You say it should be true but you do not demonstrate whether it is true. You just verify the other one by using it. How do we know it is true for k+1? That is something proven. That is what I referred to as the proof of proof.
62	**Zehra**	No, it is not something proven; it is a conjecture so taking it as it is, if it is also verified for subsequent cases, then it has a value.
63	**Ilker**	No, the correctness of induction has been proven as a method, hasn't it, sir? What do we do as a method; we do it for n=1 or say 0, for the first 2 values. Then, we assume it is true for k for n=k. If it is true for k+1 then it is true; if it is false, then it is not true. For instance, up to this point, it has been proven.
64	**Hakan**	So it means if it is true for k, it will already be true for k+1.
65	**R**	We can now return to induction; is that all you can say about proof?
66	**Yeliz**	Sir, everyone must agree on the proof of a theorem so that the proof can be true. At that point, we use propositions, lemmas, and previous things for its acceptance.

Table 4 - Excerpt 2 (Whole group discussion)

The third data group involves the semi-structured interviews conducted with the pre-service teachers. At the first stage of individual interviews, the first question addressed was again 'what is proof?' This question (Q1) aimed to determine whether there had been any change in the definitions following group discussions.

The second question was about what they associate proving with (Q2) on the basis of the response to the first question. Questioning the purposes of proofs, the third question was about what proofs provide (Q3). The results obtained in this data group are provided in Table 5.

What first captures the attention in Table 5 is that the pre-service teachers (except for Zehra, Serap and Arda) chose to express their proof definitions in a shorter and more concise form but their explanations for proving were presented in a longer fashion. Except Serap and Arda, all of them explained proof and proving separately. As had been the case in previous definitions, a majority (7 out of 8) of the definitions in Table 5 expressed proof as verification. Only (again) Yeliz did not use the term verification and defined proof as 'to arrive at the unknown from what is known'. In general, what can be said about the second definition is that they had a more formal structure when compared to the first ones.

An examination of the responses to the question about what proving provides us in Q3 shows the following (Table 5):

YELIZ	
It is to arrive at the unknown from what is known.	Q1
We have a proposition, theorem or formula; this is a process by which we gradually arrive at the unknown by making deductions on the basis of what is given through ways we already know or we can think of at the moment.	Q2
It helps me see, understand and do what is proven. Certain theorems are simple and do not require anything extra. But proving difficult theorems may help one acquire different things. First of all, it may draw us away from the practice of memorizing. Secondly, I can see the connections or relations all knowledge has a foundation.	Q3
ZEHRA	
I can say it is to demonstrate whether a given information is true or false.	Q1
To demonstrate whether a given information is true or false by grounding on a basis.	Q2
If we think of the proving process as data analysis, then the explanation is clear. Verification already requires certain bases, which show us why it is true. When I am given a statement, I first question whether it is true or false. When I prove it, I provide a logical explanation, which becomes reliable information that I can use.	Q3
BANU	
Proof is to verify something. It is to demonstrate the correctness of the data presented to us.	Q1
Proving is an operation by which we use the given data to arrive from what is given to what is desired.	Q2
First of all, I see that what is proven is true. At the school we are usually presented knowledge without proving, but we still use them. However, if it is proven, we see the steps involved.	Q3
SERAP	
It is to verify the correctness of a mathematical expression by using mathematical axioms, propositions or theorems.	Q1
It is the entire process.	Q2
It fascinates me. I can see how others reasoned by using an approach that I have never thought of. It creates a kind of awareness. Proofs allow deriving conclusions and to arrive at other results from those.	Q3
ONUR	
It is to verify the correctness of something in an unquestionable fashion.	Q1
It is to verify the correctness of a concept, statement or proposition by proceeding with certain mathematical steps.	Q2
It first inspires trust in me for that, what is done. It may offer some perspective towards phenomena. It is also a process by which I satisfy my curiosity.	Q3
HAKAN	
It is to demonstrate the correctness of a theorem or formulae.	Q1
It is to attain the correct result through mathematical operations by using the given data as well as other theorems, propositions or formulae.	Q2
First of all, I build trust for it. We can make such an analogy: If you are served a dish you sometimes eat at a different place, you would not want to eat it if it does not look nice. But if you see what is added in the dish, where and how the ingredients came from, and watch how the food was cooked, then you trust the food and eat it. It is exactly the same, trust is also built by proving.	Q3

ILKER	
It means demonstrating the existence or correctness of something.	Q1
To demonstrate that a theorem, formula or proposition is true on the basis of our existing knowledge.	Q2
If you accept that a theorem exists, and if you accept how it is proven, then you can apply it. It also helps me think multi-dimensionally and differently. Because proving is something that is done by using many different subjects, so you automatically review those subjects in your mind. Then it also helps applying it in problems. The result of a proof can only be true if the steps we take are logical in proving. And these give us the reasons, it explains them.	Q3
ARDA	
It is obtained as a result of given data; to verify the correctness of information estimated on the basis of the given data.	Q1
The way to do it; proving.	Q2
It helps seeing and thinking about critical points; you think whether there is something like that. You also acquire the way of thinking there; it explains the facts for us. It guides us in such different questions.	Q3

Table 5 - Pre-service teachers' responses to Q-1,2,3

Yeliz said it helps one to see, understand and do, and also to draw one away from the practice of memorizing by showing the connections or relations;
Zehra said it ensures trusting in data, and makes information useable, and it also offers explanations;
Banu said it demonstrates the correctness of knowledge, and helps seeing the steps involved;
Serap said it creates fascination, and the attained results may provide a basis for other results;
Onur said it builds trust for what is done, helps to develop perspective and satisfies one's curiosity;
Hakan said it builds trust for what is proven,
Ilker said it helps one to apply the information in different ways, stimulates multi-dimensional thinking, and provides explanations about reasons; Arda said it helps one to see critical points, acquire the way of thinking in the process, and offers explanations about reasons.

In the light of these opinions, the pre-service teachers apparently focused on two main objectives of proof, which are verification and explanation. Both in their first and second definitions, nearly all of the pre-service teachers stressed verification and referred to trust, convincing and seeing in Q3, which are points that highlight verification in their conceptions. Furthermore, several (Zehra, Ilker and Arda) mentioned the explanatory function of proof, which is the secondary objective. During the interviews with regard to explanations, Zehra said "verification already requires certain bases, which show us why it is true"; Ilker said "and these give us the reasons; it explains them" and Arda said "it explains the facts for us". Moreover, the statement "show the connections or relations of theorems" (Yeliz), "seeing the steps involved in proving" (Banu) and "helping see critical points" (Arda) are different kinds of responses to the question *why* *proof is true*, and each refers to explanation.

One of the questions addressed during the second stage (two weeks after the first) of the individual interviews also involved the purposes of proving. This question is referred to as Q4 in the article. Involving two sections, Q4 is as follows: Why do we prove? What is/are the purpose(s) of proving as a mathematics teacher (MT) and as an undergraduate student (US)? The two-pronged question was intended to stimulate the pre-service teachers to develop their responses from a broader perspective. Table 6 was constructed for Q4. It is clear from the table that, thinking from the perspective of a MT, the pre-service teachers explained the purposes of proving in three ways. The first was about the functions of proving, the second about the effects of proving in teaching school mathematics, and the third about its contributions for a mathematics teacher. The first group referred to reliability, presenting correct knowledge, convincing (Banu, Hakan, Ilker, Arda), demonstrating the logic behind proving (Zehra), and developing insight into the structure of mathematics (Yeliz, Zehra, Serap). The second group involved avoiding learning by memorization (Zehra, Ilker), ensuring retention of knowledge/learning (Onur, Hakan), encouraging students to engage in higher-order thinking (Ilker), helping them establish cause-and-effect relationships (Zehra, Onur), and go deeper into mathematics (Arda). The third group, on the other hand, was about improving a teacher's own mathematical perspective (Banu) and way of thinking (Onur). Among these three groups, the participants' statements about the first (7 individuals) outnumbered the statements about the second (5 individuals) and the third (2 individuals) groups. This is a natural result originating from the perception of pre-service teachers about the main function of proof as verification.

The contributions of proof to teachers' cognitive skills were only noted by two individuals, indicating that this opinion is not among the main responses to Q4.

YELIZ	
-- It can show that mathematics does not emerge in a vacuum, but can be proven. -- It can ensure consistency between some knowledge by looking scientifically at mathematics.	MT
-- Mathematics is in fact an independent abstract world dealing with proofs. I believe mathematics without proof would be much like simple market mathematics. -- I think we should carry out proofs to rise above a certain level. -- It changes the way one thinks. One should progress step by step in proving, is it not the same in life? -- I always see its contributions from a practical perspective. We can think of it as both a skill and perspective.	US
ZEHRA	
-- It can show students the logic behind what is done. -- It takes one away from the practice of memorizing in learning. -- It helps to establish cause-and-effect relations.	MT

-- To acquire academic research skills. -- Proof requires higher-order thinking; to learn this. -- To look at phenomena by establishing cause-and-effect relationships.	US
BANU	
-- To expand one's own mathematical perspective. -- To build students' trust.	MT
-- To see the origin of what is learned. -- To learn reasoning and inference.	US
SERAP	
-- To demonstrate the origin of a subject or information when students ask about them.	MT
-- Proving at undergraduate level is to prove what is already proven, right? There is nothing new that is proven. Those proofs are needed as the person specializes in that area. S/he should go deeper into mathematics and know the origin of anything. Otherwise s/he will have to memorize.	US
ONUR	
-- To help students establish cause and effect relationships. -- To develop one's own way of thinking. -- To ensure retention of information for students.	MT
-- To digest what is learned. -- To learn about the origin of knowledge.	US
HAKAN	
-- To ensure that students believe in what is done. -- To offer reliable and correct information. -- To easily convince students in what s/he teaches them. -- To ensure retention of learning.	MT
-- To go deeper into mathematics. -- To learn to be patient when studying mathematics. -- Proving helps learning to think systematically since it has a system.	US
ILKER	
-- To drive students away from memorizing. -- To help them with higher-order thinking. -- To convince them of the correctness of information.	MT
-- To improve mathematical knowledge and skills. -- *But the learning level should not be the same for students in mathematics departments in faculties of science, and those in mathematics teaching departments of education faculties. Less is needed for teaching. And they will not use proving much in high school.*	US
ARDA	
-- To show students that the science of mathematics is correct and reliable. -- To attract students to mathematics.	MT
-- To enjoy oneself. -- To learn how proving is done. -- *But not much is needed in teaching. After all, the university student will become a teacher, and proving is not often used in high school.* -- It helps multi-dimensional thinking; improves skills. Proofs are difficult and of the highest order, and one who can do it can also do many other things.	US

Table 6 - Pre-service teachers' responses to Q-4

When thinking as a US, the pre-service teachers considered that the purposes of proving as including improved comprehension regarding what mathematics is to go deeper in mathematics (Hakan, Serap), as well as acquisition of knowledge and skills. The skills in question were listed as higher-order thinking skills (Yeliz, Zehra), the ability to think differently and multi-dimensionally (Yeliz, Arda), academic research skills (Zehra), reasoning and inference (Banu), learning to be patient, and thinking systematically when studying mathematics (Hakan), and learning how to carry out proofs (Arda). The fact that seven out of eight people mentioned skills revealed that the opinion was embraced by a majority of the participants. In addition, Ilker and Arda commented on the way proofs are taught at mathematics departments and education faculties. Both pre-service teachers had the same idea, arguing for less content about proofs in the curriculum of education faculties, which they considered unnecessary becasue teachers do not often teach proofs at the secondary school level.

5 Discussion

As is reported in the literature, undergraduate students as well as students at different levels often have problems with proof (WEBER, 2004). Studies have reported various reasons for such problems (MARTIN; HAREL, 1989; KNUTH; ELLIOTT, 1997; KNUTH, 2000; WEBER, 2001). One of these reasons is the insufficient level of conceptions regarding proof among learners. 'Conception of proof' is considered here in a broad sense, and can be addressed in its various dimensions, such as learners' knowledge of proof, beliefs about importance, nature and roles or purposes of proof, and understanding of proof.

In the study, examining the participants' definitions of proof and opinions about the purposes of proof, *verification* was the concept highlighted in the first definitions presented in response to the first research question. Drawing upon common points of these definitions, it could be suggested that the participants defined proof as "*the verification of a mathematical statement*". The second point addressed in the research was to what extent the participants conceive the resulting definition. Group discussions provide some data regarding this point. In Excerpt 1, Onur asked whether proving is necessarily done by using things that are known (19), Zehra mentioned that proving can simply be made by reasoning (21), Zehra also mentioned induction among the premises used in proofs (32) and by saying "not every proof requires a proposition (38)", she claimed [*meant*] that what is required is some previous knowledge such as proving methods and mechanism since proving is a higher-order skill; in Excerpt 2, Arda stated that he could not understand how induction can be a method of proof (42) and Ilker characterized induction as the proof of proof (51). Excerpts 1 and 2 show us that there are some problems in participants conceptions. It can be observed that Onur and Zehra were not able to grasp sufficiently the necessity of some premises (axioms, propositions, theorems, etc.) in the proving process. Zehra described induction as a premise; however, it is a method of proof, not a premise. Likewise, the questions Arda and Ilker have on their minds also reveal some confusion. Moreover, Ilker's description of proof as 'the proof of proof' is both interesting and incorrect. Yet, it is known that these problems are not simply limited to the participants of this study. For instance, Reiss et al. (2008) reported that most students have problems in the transitions between inductive and deductive reasoning in mathematics.

The pre-service teachers were asked to define proof and proving once again during the interviews. In this context, most of the proof definitions assumed a shorter and more concise form, while their definitions of proving contained longer and additional statements. These additions were as follows: "by ways we can think of at the moment (Yeliz)", "through certain bases (Zehra)", "by proceeding with certain mathematical steps (Onur)", "through mathematical operations (Hakan)", "estimated on the basis of the given data (Arda)". These expressions revealed that the participants were particularly careful about the aspect of mathematical steps by slightly separating the process of proving from proof. Participants tended to provide a more formal definition, but formality here does not mean rigorous[2]. This result demonstrates that the participants changed their ideas somewhat when addressing proof as a concept versus a process. In the definitions, proof indicates the basic purpose of what is done, while proving focuses on its mechanism.

Another point to make is that definitions presented in the individual interviews were more formal compared to the written definitions, if we think of proof and proving together. Still, even the conceptions presented in the interviews fail to refer to the systematic based on logical arguments, mathematical notation and language, and axiomatic and symbolic manipulation. Therefore, the participants´ conceptions are arguably far from the formal (rigorous) or logical proof definition shown and exemplified in their departments in perceptual terms, even though they graduated from mathematics departments and had seen many proofs before. Despite the fact that the participants were not specifically asked to give a definition of rigorous proof, it was possible for them to have mentioned some points, considering the way proofs are presented in mathematics departments in Turkey. It is an expectation that stems from their education, not the author's own expectation. Such points include the requirement that a certain type of language and notation be used, that the steps in a proof are logically connected to one another, and defining proof in the propositional logic (when a theorem [$p \rightarrow q$] is given, the proof of this theorem is the demonstration that q is true on the condition that p is true) and so on. In a study, Knuth (2002a) attempted

[2] *Rigorous proof* as a concept that is employed mostly by formalists, it emphasizes the fact that proof structurally has a fairly strict form in an axiomatic system.

to determine the conceptions of 17 secondary school mathematics teachers, and reported that 9 of the participants defined proof in a formal fashion, according to a categorization system of formal, less formal and informal. Zhou and Bao (2009) investigated the proof competence of China's Masters-level secondary school mathematics teachers (152 individuals in total). The teachers were asked the question "what is a mathematical proof?", and the authors reported that 80% of the participants' responses were categorized under "*proofs must be based on logical axioms, previously proven theorems, or propositions*" and 86% emphasized that "proofs should follow logical rules". To some extent, the results of both studies corroborate those of the present study with regard to formality. However, Zhou and Bao (2009) stated that the participants in their study gave answers using specialized vocabulary (deducting step by step, rigor, evidence for every sentence, etc.), and this is what distinguishes their results from this study.

The second research question examined the participants' main opinions about the purposes of proof. As for Q3, two main perceptions were found among the participants. The first and most emphasized purpose cited by the participants is to validate or verify the truth of a statement, while the second is the explanatory function. The participants' opinions changed when the same question was addressed in Q4 from two different perspectives (MT and US). They expressed opinions in three ways for MT (the functions of proof, its effects on teaching school mathematics, and its contributions to mathematics teachers); but when it came to US, most of them mentioned the knowledge and skills acquired in the process of proving. In response to Q3, Ilker (multi-dimensional thinking) and Arda (acquiring the way of thinking in the proving process) mentioned skills, while in response to Q4, five individuals (Yeliz, Zehra, Arda, Banu, Hakan) referred to skills using a richer categorization. Furthermore, the opinions in Q3 simply referred to mathematics, but those in Q4 referred to both mathematics and mathematics education. Although the main function of proof cited in the pure mathematics and some research literature is said to be verification (DE VILLIERS, 1999; HANNA, 1995; SCHOENFELD, 1994); in fact, there is now a consensus on the fact that proof has greater functions both in terms of mathematics and mathematics education (YACKEL; HANNA, 2003; KNUTH, 2002a; 2002b; DE VILLIERS, 1999). Thus, while the participants' main perceptions about the purposes of proof expanded, they overlapped with two of the functions mentioned in the literature.

Varghese (2009) investigated the conceptions of 17 student teachers who were in the final semester of a teacher education program at a large Canadian university. At the first stage, he asked the participants to define "the notion of proof". According to the predominant opinion (9 individuals) proofs signified "verification". Other purposes were only mentioned by one or two individuals. As a sequel to a previous study Knuth (2002a), Knuth (2002b) examined the conceptions of 16 mathematics teachers in terms of 'roles of proof in mathematics'. The purpose with the highest amount of participant agreement was "communication of mathematics (2)", which was followed by "creation of knowledge/systematization of results (8)". "Establishment of truth (4-6)" and "explanation (0-3)" appeared in multiple sub-categories, with each category mentioned by less than six. How proof was perceived by the teachers (formal, less, informal) also influenced the distribution of these figures.

The final research question investigated whether the opinions revealed in the first and second questions could change. As was demonstrated in the previous sections, the opinions about both proof (and proving) and the purposes of proof were mutable. A change occurred in the participants' perceptions depending on the time, group discussions and the perspectives elicited in the questions. Rather than a radical change or a kind of cleavage, this change was characterized more by expansion and integration. Likewise, in Knuth's 2002a and 2002b studies, the teachers were able to present their perceptions (supplementary) from different perspectives when addressing proof in the context of mathematics and secondary school mathematics. The present study found that rather than contexts, only the opinions obtained at different times and by different means had an expanding and integrating character.

In the light of the participants' definitions of proof and opinions regarding the purposes of proving, the study demonstrated that their conceptions about proof are limited and reveal a narrow perspective.

6 Further remarks

The pre-service teachers participating in the study were graduates of mathematics departments in five different public universities in Turkey. Despite being educated in different universities, their conceptions about proof and its purposes did not differ radically. Even though a positive change was detected in their second opinions, it can still be observed that their conceptions remained limited to a narrow perspective. One of the main reasons behind this is the limited emphasis given to proofs in (secondary and high) school mathematics curriculums in Turkey, and the fact that proofs are often taught in mathematics courses at mathematics departments in universities in a direct way. What Knuth (2000) suggests is significant with regard to the first reason. He argues that students do not get enough exposure to proofs in school mathematics, and it is therefore not surprising that studies reveal problems with respect to their understanding of proof. As for the second reason, it will be useful to refer to the participants' discourses.

> I graduated from a faculty of science and letters, and let me tell you this. We were given theorems. We studied them and took exams on them. But we have proven hardly any theorems by ourselves. We had a few teachers who would do that (Zehra).

> I wish we had been given opportunities to think about it; it would be much different then, but we were not. When they asked about proofs, we simply came to class after memorizing them (Banu).

> Personally speaking, I suffered from proofs. They would show us a lot of proofs in succession, so we had to memorize (Onur).

Nevertheless, university mathematics constitutes the main source for the content knowledge of mathematics teachers (JONES, 2000). Therefore, the university education of pre- and in-service teachers strongly affects their conceptions about proof and proving, an effect which should not be overlooked. In this respect, Harel and Sowder (2007) noted that they find the research on teacher conceptions of proof noteworthy since they draw the attention of mathematics departments which bear the primary responsibility for preparing students.

The present study contributes useful information through an approach that identifies participant opinions regarding the definition of proof and its purposes, as well as changes in these opinions. Thus, the same approach is recommended for further research.

Furthermore, given the limited amount of research on the conceptions of pre- and in-service mathematics teachers in Turkey, there is an obvious need for further research in the field.

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